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Math 636 – Mathematical Modeling

Homework #2

31 August 2021

Problems:

32: 2, 3, 12

34: 3, 7, 14

**Problem Section 32.2:**

Suppose that the birth rate of a species is 221 per 1000 (per year), and the death rate is 215 per 1000 (per year). What is the predicted population as a function of time if the species numbers 2000 in 1950?

We know from page 123 of the text that the birth rate is Also,

Given that our populations are described in years, our which is a measurement from year to year.

Plugging this into gives (assuming is a constant):

Hence, our difference equation becomes (from page 124):

Here, if we successively plugged the results of this equation back into itself, we would get:

Now, if we wanted to show this as a function with our specific initial conditions above; namely , , then we would have:

**Problem Section 32.3:**

Suppose in addition to births and deaths (with constant rates and respectively), that there is an increase in the population of a certain species due to the migration of 1000 individuals in each interval of time.

**a)** Formulate the equation describing the change to the population.

We would now adjust the equation to reflect that during each cycle, an additional 1000 individuals enter the population.

This is what we expect to see from our already discretely formulated solution when we simply add 1000 individuals per time interval.

**b)** Explicitly solve the resulting equation. Assume that the initial population is .

Here we note that . Starting at and again noting that :

Now at :

At :

Here, we can see a pattern emerging. So, we can write this formula as:

**c)** Verify that if , your answer reduces to the correct one.

Here we note first that if the birth rate and the death rate are the same, and we add 1000 individuals per discrete , then what we should have will simply be some initial population, , gaining 1000 population per cycle, and hence we’ll obtain the linear equation:

Here if , then obviously our equation equals out to . Instead, observing our equation directly we note that if then So:

Adjusting our sigma notation simply gives us:

Lastly, we can again start at and , so:

This confirms our initial thoughts on how this system would proceed if the birth and death rates were the same with an influx of a fixed number of new individuals.

**Problem Section 32.12:**

Suppose that you borrow dollars (called the principle) from a bank at percent yearly interest and repay the amount in equal monthly installments of dollars.

**a)** If is the money owed at time , show that and

What is and ?

Here we know that is 1 month, i.e. the monthly payments, so . We see also that that , given that is the yearly interest (this would be 1 if monthly). Manipulating units and keeping in mind our is months not years:

Hence, when is converted from years to months it becomes . So our model becomes:

Focusing on just the positives (the things that increase our amount owed) we have:

So:

**b)** For example, part of the first payment consists of interest. How much of your first payment goes towards reducing the amount owed?

The monthly payment that you make, , is “lessened” by the amount you have to pay in interest. Therefore, if we let be the amount reduced by the first payment, i.e. how much actually goes towards paying off that debt, then (here we assume that the payment is larger than the accrued interest):

**c)** Solve the equation for .

Handling the first term utilizing our equation above

Acknowledging that and we’ll set meaning that adding increases the amount owed at discrete time intervals of .

So, we’ll now plug in successive iterations into this equation, starting with :

Now for

At

We can see a clear pattern emerging now that gives us the following expression:

**d)** How much should your monthly payments be if the money is to be completely repaid to the bank in years?

Here, we’re effectively asking, after a specific amount of years, i.e. discrete time steps, will it take for using a specific monthly payment . Using our equation above we’ll set , and (this becomes because is in years which we convert to months):

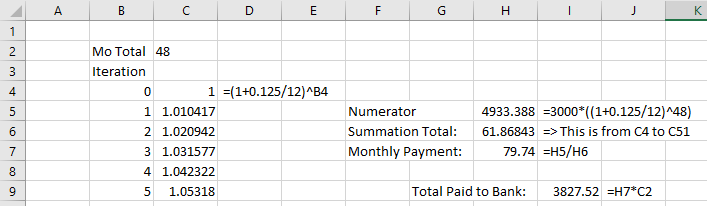
**e)** What is the total amount of money paid to the bank?

From the above equation in part d) we found how much it would cost per month to pay off the loan over the course of amount of months. Therefore, however long the loan is repaid for, times the amount paid each month, i.e. , is the total paid to the bank.

**f)** If you borrow for a car and pay it back monthly in 4 years at 12.5 percent yearly interest, then how much money have you paid in total for the car?

Using our formula from above (:

Performing the calculations in MS Excel for , then doing .

Work done in Excel: 

**Problem Section 34.3:**

The growth rate of a certain strain of bacteria is unknown, but assumed to be constant. When an experiment started, it was estimated that there were about 1500 bacteria, and an hour later 2000. How many bacteria would you predict there are four hours after the experiment started?

Here we have , and Also, we’re given that We also note that because the population is fairly large and grows quickly, we’ll use a continuous model rather than a discrete one. Now, modeling our bacteria:

Solving:

Plugging in our values for :

So, we have:

Now, utilizing our second condition at :

Performing this basic calculation gives us:

This means our model now is:

The question asks for the population at which we can now easily calculate:

**Problem Section 34.7:**

Consider a species which is modeled as growing at an instantaneous rate of 3 percent per year. Another species grows at 3 percent per year when measured every year. Compare the time it takes both species to double.

Here we have the difference between a discrete and a continuous model of growth. Our discrete model gives us (noting that is the iterations and is a continuous measurement of time):

And the continuous model gives us:

If we assume that for year time steps, and use the given that , we can rewrite our two equations as:

Disc.

Cont.

Solving both for their respective measurements of time after doubling the population:

Disc.

Cont.

Clearly, we can see that the continuous case has a slightly faster time to double. Therefore, we can conclude that continuous 1D models will have faster growth with exponential growth at fixed rates.

**Problem Section 34.14:**

The parameters of a theoretical population growth curve are often estimated making the best fit of this curve to some data. If discrete population data is known (not necessarily measured at equal time intervals), , then the mean-square deviation between the data and a theoretical curve , is

the sum of the squared differences. The “best” fit is often defined as those values which minimize the above mean-square deviation.

**a)** Assume that the initial population is known with complete certainty, so that we insist that the theoretical population curve initially agree exactly. Assume the theoretical curve exhibits exponential growth. By minimizing the above mean-square deviation, obtain an equation for the best estimate of the growth rate. Show that this is a transcendental equation.

First, we’ll rewrite the population as a function of time inside of our summation:

Then, we’ll take the partial derivative with respects to which should give us the optimized value for our constant rate of change (we also note that at our values of are simply constants):

Now, we set this to zero and try to solve for (in the special case where we note that the summation is equal to zero and hence the statement is true, so the following is for all such that ):

We’re finally left at a bit of an awkward point. Clearly the above expression cannot be evaluated easily terms of . Hence, what we’re left with is a transcendental equation that could be used to solve for (as we know and all of our various data points), but some type of numerical methods would be required to actual obtain a value for .

**b)** One way to bypass the difficulty in part (a) is to fit the natural logarithm of the data to the natural logarithm of the theoretical curve. In this way the mean-square deviation is

Show that this method now is the least squares fit of a straight line to data. If is known , determine the best estimate of the growth rate using this criteria.

The first observation we can make is that we know (as ) the following:

Assuming that for our relative time measurements, then:

Observing the summation above equation, we see it takes the form of , where and . In this example, we know what our “y-intercept” (which here would be our population intercept at time zero) is, so the resulting slope of the line would be .

Minimizing with respects to :

Now we set this to zero again and note one more time we’re excluding the case of :

Therefore, with simply being some known constant, we can rearrange this equation to be:

As desired, we see that now we no longer have a transcendental function for after linearizing the data using the natural log function.

**c)** Redo part (b) assuming that a best estimate of the initial population is also desired (i.e. minimize the mean-square deviation with respect to both and ).

Here our isn’t a fixed point as it was previously. Therefore, we’ll once again find but also with the assumption that at we don’t know . Starting with :

Utilizing our work from part b) for gives us the below equation. However, we note now that is a variable instead of a constant now:

Solving for :

Now for (excluding the trivial case of ):

Now, we see that we have two equations and two unknowns, meaning we can solve the system of equations. Plugging in our expression for above:

Or rearranging the top terms to be nicer:

(For calculation purposes we’d have the following:

but we’ll keep the summation notation for easier plugging back into of .)

Now placing this all back into the equation to find :

We’ll now solve this symbolically to make the rearrangement of terms simpler (each unique summation is assigned a corresponding letter):

Replacing our symbolic lettering again with the appropriate values and summations (and listing below for completeness):

We can then utilize this information to generate a best fit model to .